Scaling of the magnetic permeability at the BKT transition

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We revisit the Berezinskii–Kosterlitz–Thouless (BKT) transition from the perspective of magnetic permeability. By allowing non-neutral vortex configurations in a Coulomb gas model, we uncover multiplicative logarithmic corrections to scaling governed by a dangerously irrelevant fugacity. The resulting scaling is unexpectedly simple, yet not commonly emphasized.

By 2017, one might have reasonably believed that the Berezinskii–Kosterlitz–Thouless (BKT) transition had already revealed all its secrets. After all, it is one of the most celebrated results in statistical physics: a topological phase transition driven by the unbinding of vortex–antivortex pairs, with a well-established renormalization group (RG) description and a long list of experimental realizations [1, 2].

And yet, perhaps out of curiosity (or stubbornness), we wondered whether there was still something left to say. The usual approach to the BKT transition focuses on quantities such as the superfluid stiffness or the dielectric response. These observables work extremely well, and their scaling properties are by now textbook material. So naturally, professors Mats Wallin and Jack Lidmar, from KTH in Stockholm, proposed to look at something else: the magnetic permeability μ [3].

At first glance, this may not seem like the most promising idea. In the standard Coulomb gas formulation with periodic boundary conditions, the total vorticity is strictly zero, and therefore the permeability vanishes trivially. Not an encouraging starting point.

The way out was simple: allow the system to fluctuate. By introducing a screening length $\lambda \sim L$, proportional to the system size, one permits non-neutral vortex configurations while preserving the BKT transition in the thermodynamic limit.

Within this framework, the system can be described as a two-dimensional Coulomb gas, where vortices play the role of charges interacting via a logarithmic potential. In this representation, the dielectric response characterizes how these charges screen each other, while the magnetic response is related to fluctuations of the total vorticity. The two descriptions are therefore closely connected, providing complementary perspectives on the same underlying physics.

The key observation is that each vortex carries a quantized unit of magnetic flux. As a result, the total vorticity m is directly proportional to the magnetic flux through the system, establishing a bridge between the Coulomb gas description and the magnetic response of a thin superconducting film. In this way, the compressibility of the Coulomb gas becomes equivalent to the magnetic permeability.

Up to non-universal constants, this leads to

$$\mu \propto \frac{1}{T}\left(\langle m^2\rangle - \langle m\rangle^2\right). \tag{1}$$

So far, so good. The next step was to ask how $\langle m^2\rangle$ scales with system size. A naive argument suggests that at criticality the fluctuations should be size-independent,

$$\langle m^2\rangle \sim L^0. \tag{2}$$

Had this been correct, the story would have ended there. But this is the BKT transition, where nothing is ever quite that simple.

The correct scaling behavior follows from the RG equations of the Coulomb gas. Introducing reduced variables x and y for temperature and vortex fugacity, the flow equations are

$$\frac{dx}{dl} = 2y^2, \qquad \frac{dy}{dl} = 2xy, \tag{3}$$

with invariant

$$x^2 - y^2 = C^2. \tag{4}$$

At and below the transition, the fugacity flows to zero. One might therefore be tempted to ignore it. This temptation should be resisted. This is a clear instance where naive scaling fails due to a dangerously irrelevant variable.

Although the fugacity vanishes under coarse graining, it still controls physical observables through multiplicative factors. In particular, the leading contribution to $\langle m^2\rangle$ is proportional to y, making explicit that the scaling is governed by the renormalized fugacity:

$$\langle m^2\rangle \sim y(L). \tag{5}$$

Solving the RG equations at the transition yields a slow logarithmic flow,

$$y(L) \sim \frac{1}{\ln(L/L_0)}, \tag{6}$$

and therefore

$$\langle m^2\rangle^{-1} \sim A\ln L + B. \tag{7}$$

Remarkably, this form provides a direct signature of the dangerously irrelevant fugacity, turning what would otherwise be constant fluctuations into a logarithmically vanishing quantity. This behavior stands in

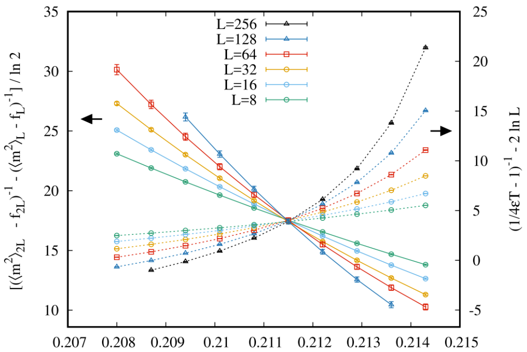

Figure 1: Intersection plot for the vorticity fluctuations including second-order logarithmic corrections (solid lines, left axis). Data constructed from pairs of system sizes $(2L, L)$ collapse into a sharp crossing at the BKT transition. For comparison, the corresponding intersection curves for the dielectric response are also shown (dashed lines, right axis), exhibiting a consistent estimate of the transition temperature.

sharp contrast with naive power-counting, which would predict constant fluctuations, and highlights how seemingly negligible variables can leave measurable imprints on macroscopic observables.

From a physical point of view, this result reflects the role of rare, unbound vortices. The fugacity controls the probability of creating such excitations, and although this probability vanishes under renormalization, it remains the only mechanism capable of producing net vorticity fluctuations. In this sense, the observable is dominated by events that become increasingly rare, yet never entirely disappear.

Numerical results. The asymptotic form in Eq. (7) provides a first indication of the critical behavior, but finite-size data require the inclusion of subleading corrections. These corrections arise naturally within RG and can be incorporated as

$$\langle m^2 \rangle = \frac{1}{A \ln L + B + C \frac{\ln \ln L}{\ln^2 L}}. \tag{8}$$

Rather than fitting this expression directly, it is more effective to construct combinations of data for pairs of system sizes $(2L, L)$, eliminating non-universal constants and exposing the scaling structure more transparently. This leads to a size-dependent quantity whose curves intersect at the critical temperature, providing a direct and robust estimate of T_c.

The result of this construction is shown in Fig. 1. For comparison, the corresponding curves for the dielectric response are also included, highlighting the close relation between charge fluctuations in the Coulomb gas and the magnetic response of the system.

The intersection is clean and well-defined, yielding a precise estimate of the transition temperature. In a practical sense, this provides a direct way of identifying the transition, without relying on delicate fitting procedures.

What is perhaps more satisfying is how directly the expected scaling behavior emerges once the appropriate corrections are taken into account. Rather than being obscured by finite-size effects, the logarithmic structure predicted by RG becomes clearly visible in the data, almost as if the scaling structure were directly exposed by the system itself.

It is worth contrasting this behavior with more conventional observables used to characterize the BKT transition, such as the superfluid stiffness or the dielectric response. In those cases, the transition is identified through universal properties or discontinuities, and finite-size effects tend to obscure the asymptotic scaling. By contrast, the vorticity fluctuations provide a more direct probe of the underlying RG flow, making the logarithmic structure visible in the particularly clear way of Fig. 1.

This makes the permeability-based approach not only viable, but also particularly intuitive. The scaling is simple, the numerical implementation is straightforward, and the resulting behavior is unusually clear—a rare combination in the study of critical phenomena.

Below the transition, the scaling crosses over to a power law, $\langle m^2 \rangle \sim L^{2x(T)}$, with $x(T) \approx 1 - 1/4T$, in agreement with RG expectations.

From an RG perspective, the origin of this behavior lies in the structure of the fixed line below the transition, where the fugacity flows to zero while the temperature approaches a finite renormalized value. The vanishing of the fugacity suppresses vortex creation, but at the same time introduces a scale dependence that modifies naive scaling laws. The result is a multiplicative correction that cannot be captured by simple power counting.

More broadly, this example illustrates a general mechanism in critical phenomena: variables that are formally irrelevant in the RG sense can nevertheless control observable scaling through multiplicative effects. The fugacity in the BKT transition provides a particularly transparent realization of this idea.

Similar mechanisms appear in other contexts, such as ϕ^4 theory above its upper critical dimension, where interactions become irrelevant yet remain essential for determining fluctuations. In this sense, the BKT fugacity is not an exception, but a particularly clear realization of a more general principle.

It is perhaps somewhat surprising that such a simple and transparent scaling behavior has not been more widely explored.

Notes

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